Phys 410
Fall 2015
Lecture #22 Summary
12 November, 2015

We are considering the description of Newton's second law from the perspective of an observer on the rotating object. An observer in the "body frame" can identify the principal axes of the object and use them as a coordinate system to describe the angular momentum using the diagonalized inertia tensor as $\vec{L} = (\lambda_1\omega_1, \lambda_2\omega_2, \lambda_3\omega_3)$. One can write the equations of motion as witnessed in the body frame as $\vec{\Gamma} = \left(\frac{d\vec{L}}{dt}\right)_{Body} + \vec{\omega} \times \vec{L}$, which translates in component form into the Euler equations:

$$\Gamma_1 = \lambda_1\dot{\omega}_1 - \omega_2\omega_3(\lambda_2 - \lambda_3)$$

$$\Gamma_2 = \lambda_2\dot{\omega}_2 - \omega_1\omega_3(\lambda_3 - \lambda_1)$$

$$\Gamma_3 = \lambda_3\dot{\omega}_3 - \omega_1\omega_2(\lambda_1 - \lambda_2)$$

The hard part of using these equations is to translate the description of the torque $\vec{\Gamma}$ as witnessed in the inertial space frame into its projections on to the (continuously rotating) body frame axes (yielding Γ_1, Γ_2, Γ_3). To simplify things further we take the special case of rotational motion under torque-free conditions, $\Gamma_1 = \Gamma_2 = \Gamma_3 = 0$. An application of these equations was to the case of an object rotating about one principal axis ($\hat{e}_3$), but then given a small kick to produce rotations about the other axes ($\hat{e}_1$). The analysis led to a simple equation of motion: $\ddot{\omega}_1 = -\left[\frac{(\lambda_3-\lambda_2)(\lambda_3-\lambda_1)}{\lambda_1\lambda_2}\omega_3^2\right]\omega_1$, where it was found that ω_3 is approximately constant. This yields simple-harmonic motion (SHM) for ω_1 and ω_2 as a function of time if λ_3 is either the largest or smallest of the three principal moments. SHM means that the motion about the original axis is stable and the motion about the other axes oscillates around zero. If λ_3 is the middle eigenvalue, then the square-bracket term is negative, yielding a solution for $\omega_1(t)$ that grows exponentially in time. This is a sign of instability. We demonstrated this phenomenon with a book, where motion around the principal axes with largest and smallest moments was fairly stable, while motion about the third (middle moment of inertia) was clearly much less stable. The video from the ISS showing torque-free rotational motion of the Russian-English dictionary illustrated this quite clearly.

Next we derived the equation of motion that describes nutation of a precessing gyroscope in a gravitational field. First we introduced the Euler angles. This is a convention that describes the orientation of an object based on a combination of both the space frame and the body frame defined by the principal axes of the object. Note that there are many variations on the definitions of the Euler angles, so be very careful to choose once convention and stick to it. Starting with both coordinate systems aligned ($\hat{x} \leftrightarrow \hat{e}_1$, etc.) and the origins coincident, first rotate the rigid body by an angle φ around the $\hat{z}$ axis. Next rotate about the new $\hat{e}_2{}'$ axis by an angle θ to create the final $\hat{e}_3$ direction. Finally rotate by an angle ψ about the $\hat{e}_3$ axis. This process is simply a convention for how to align an arbitrary rigid body relative to a fixed point (the origin).

With the Euler angles, we can now express the angular velocity and angular kinetic energy in terms of these angles and their time derivatives. In other words we can say $\vec{\omega} = \dot{\phi}\hat{z} + \dot{\theta}\hat{e}_2' + \dot{\psi}\hat{e}_3$ and for the kinetic energy $T = \frac{1}{2}\vec{\omega} \cdot \vec{L} = \frac{1}{2}(\lambda_1\omega_1^2 + \lambda_2\omega_2^2 + \lambda_3\omega_3^2)$, which is written in terms of the body frame. If we translate the expression for the rotational kinetic energy in to the Euler angles, after some vector algebra, we get, $T = \frac{1}{2}\lambda_1\left(\dot{\phi}^2 \sin^2\theta + \dot{\theta}^2\right) + \frac{1}{2}\lambda_3\left(\dot{\psi} + \dot{\phi}\cos\theta\right)^2$. This expression assumes a cylindrically symmetric object (e.g. a gyroscope) with $\lambda_1 = \lambda_2$. The potential energy of the gyroscope is simply $U = MgR\cos\theta$, where M is the total mass of the gyroscope, R is the location of the center of mass from the point of support (the origin here), and g is the acceleration due to gravity. The Lagrangian for the gyroscope is just $\mathcal{L} = T - U$, and one sees immediately that both ϕ and ψ are cyclic (or ignorable) coordinates, hence their conjugate momenta p_ϕ and p_ψ are constants of the motion. These constants are equal to the z-component and the $\hat{e}_3$-component of the angular momentum vector. The remaining θ equation is a one-dimensional second order differential equation for a "particle" that lives in a limited-range effective potential $U_{eff}(\theta)$ with total energy $E = T + U_{eff}(\theta)$ given by $E = \frac{1}{2}\lambda_1\dot{\theta}^2 + U_{eff}(\theta)$ and $U_{eff}(\theta) = \frac{\left(p_\phi - p_\psi\cos\theta\right)^2}{2\lambda_1\sin^2\theta} + \frac{{p_\psi}^2}{2\lambda_3} + MgR\cos\theta$. The effective potential diverges at the two limiting values of θ, namely 0 and π, and forms a minimum in between. The "θ-particle" with finite total energy E therefore bounces back and forth between two classical turning points in θ, which represent the limits of the "nodding" up and down, which is the phenomenon known as nutation.